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# Novel Perturbation Expansion for the Langevin Equation

**Carl Bender,[1] Fred Cooper,[2] L. M. Simmons, Jr.,[3] Pinaki Roy,[4] and Greg Kilcup[5]**

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We discuss the randomly driven system $dx/dt = -W(x) + f(t)$, where $f(t)$ is a Gaussian random function or stirring force with $\langle f(t) f(t') \rangle = 2\delta(t-t')$, and $W(x)$ is of the form $gx^{1+2\delta}$. The parameter $\delta$ is a measure of the nonlinearity of the equation. We show how to obtain the correlation functions $\langle x(t)\, x(t') \cdots x(t(n)) \rangle_f$ as a power series in $\delta$. We obtain three terms in the $\delta$ expansion and show how to use Padé approximants to analytically continue the answer in the variable $\delta$. By using scaling relations, we show how to get a uniform approximation to the equal-time correlation functions valid for all $g$ and $\delta$.

**KEY WORDS**: Langevin equation; delta expansion; nonlinear; perturbation expansion; scaling relations.

## 1. INTRODUCTION

Recently a new perturbative technique, the $\delta$ expansion, was proposed to solve nonlinear problems in physics.[1-3] The technique involves replacing, in a differential equation, nonlinear terms such as $x^3$ by $x^{1+2\delta}$ and expanding this term in powers of $\delta$:

$$x^{1+2\delta} = x \sum_{n=0}^{\infty} \delta^n \frac{(\ln x^2)^n}{n!} \tag{1.1}$$

We are thus able to obtain a solution to the differential equation as a power series in $\delta$. The parameter $\delta$ is a measure of the nonlinearity of the

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[1] Physics Department, Washington University, St. Louis, Missouri 63101.
[2] Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.
[3] Santa Fe Institute, Santa Fe, New Mexico 87501.
[4] Electronics Unit, India Statistical Institute, Calcutta 700035, India.
[5] Department of Physics, Ohio State University, Columbus, Ohio 43210.



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theory. When $\delta=0$ the theory is linear and typically can be solved in closed form. As $\delta$ increases from zero, the nonlinearity turns on smoothly. Typically the $\delta$ series has a finite radius of convergence. Since we are interested in $\delta=1$, 2, etc., we need a way of analytically continuing the series obtained to large $\delta$. To do this, we will employ Padé approximants.[4] The first nontrivial Padé approximant, the [1, 1] Padé, requires calculating terms up to order $\delta^2$ in this expansion. We will also utilize a scaling argument to obtain the correct functional dependence of the correlation functions on the coupling constant $g$ for all values of $\delta$.

In this paper we will be studying the one-dimensional Langevin equation

$$\frac{dx}{dt}=W(x)+f(t), \qquad W(x)=gx^{1+2\delta} \tag{1.2}$$

[For this equation to be well defined for arbitrary $\delta$ and negative $x$ we interpret $W$ as follows: $W(x)=gx(x^2)^{\delta}$.]

The stirring force $f(t)$ is a random function described by Gaussian statistics, i.e., it is described by a joint probability functional

$$P[f]=N\exp\left[-\frac{1}{2}\int_{t_0}^{\infty} dt\,dt'\,f(t)\,S(t,t')\,f(t')\right] \tag{1.3}$$

with

$$\int P[f]\,\mathscr{D}f=1$$

Choosing white noise,

$$S^{-1}(t,t')=2\delta(t-t')$$

we have that

$$\langle f(t)\,f(t')\rangle=\int \mathscr{D}f\,P[f]\,f(t)\,f(t')=S^{-1}(t,t')=2\delta(t-t') \tag{1.4}$$

where $\mathscr{D}f$ denotes functional integration.

There are two ways to determine the correlations in $x(t)$ resulting from the statistics of the forcing term. One way is to solve directly for $x(t)$ in terms of $f(t)$ and then use (1.4). The other is to make a change of variables in the functional integral (1.4) to obtain a path integral in the variable $x(t)$. One has[5,6]

$$\langle x(t)\,x(t')\rangle_f=\int \mathscr{D}f\,P[f]\,x(t)\,x(t')=\int \mathscr{D}x(t)\,P[f(x)]\,x(t)\,x(t')\det\left|\frac{\delta f}{\delta x}\right| \tag{1.5}$$

Because of the retarded boundary conditions, one has that[5]

$$\det \left| \frac{\delta f}{\delta x} \right| = \exp \left[ -\frac{1}{2} \int_{t_0}^{\infty} \frac{\delta W(x(t))}{\delta x(t)} \, dt \right] \tag{1.6}$$

Thus, using (1.2) and integrating by parts, we obtain the following functional integral for the two-point correlation function[6]:

$$\langle x(t)\, x(t') \rangle_f = \int [\mathscr{D}x(t)]\, x(t)\, x(t') \exp\{-S[x]\} \tag{1.7}$$

where

$$S[x] = \int_{t_0}^{\infty} \left[ \frac{1}{4} (\dot{x})^2 + \frac{1}{4} W^2 - \frac{1}{2} W' \right] dt \tag{1.8}$$

We recognize this functional integral as the Euclidean path integral for a supersymmetric quantum mechanical system[7] when $t_0 \Rightarrow -\infty$. Thus when $t, t' \Rightarrow \infty$ with $|t - t'|$ held fixed the correlation function (1.7) becomes that for the related quantum mechanical system.

The equal-time correlation functions for this system at large times, $t \gg t_0$, can be obtained directly from the time-independent solution of the related Fokker–Planck equation.[8] If we define

$$\tilde{P}(z) = \langle \delta(z - x(t)) \rangle_f \tag{1.9}$$

then we can show that $\tilde{P}$ obeys the following equation:

$$\frac{\partial \tilde{P}}{\partial t}(z) = \frac{\partial}{\partial z} [W(z)\, \tilde{P}(z)] + \frac{\partial^2 \tilde{P}}{\partial z^2}(z) \tag{1.10}$$

and that in the steady state

$$\tilde{P}(x) = N \exp \left[ -\int^{x} W(y)\, dy \right] \tag{1.11}$$

and

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} dx\, x^n \tilde{P}(x)}{\int_{-\infty}^{\infty} dx\, \tilde{P}(x)} \tag{1.12}$$

We will use the fact that the equal-time correlation functions are exactly determined from (1.12) to make a detailed numerical study of the accuracy of the $\delta$ expansion.

## 2. $\delta$ EXPANSION FOR THE LANGEVIN EQUATION

For $W(x) = gx^{1+2\delta}$, the Langevin equation is

$$\frac{dx}{dt} + gx^{1+2\delta} = f(t) \tag{2.1a}$$

The $\delta$ expansion of this equation is

$$\frac{dx}{dt} + gx \sum_{n=0}^{\infty} \frac{\delta^n (\ln x^2)^n}{n!} = f(t) \tag{2.1b}$$

To solve this for $x(t)$ in a $\delta$ expansion, we assume that $x(t)$ can be written as

$$x = x_0 + \delta x_1 + \delta^2 x_2 + \cdots \tag{2.2}$$

Inserting (2.2) in (2.1b), we obtain a sequence of linear equations:

$$\frac{dx_0}{dt} + gx_0 = f \tag{2.3a}$$

$$\frac{dx_1}{dt} + gx_1 = -gx_0 \ln x_0^2 \tag{2.3b}$$

$$\frac{dx_2}{dt} + gx_2 = -2gx_1 - gx_1 \ln x_0^2 - \frac{1}{2} gx_0 (\ln x_0^2)^2 \tag{2.3c}$$

and so on. [Notice that if $W(x)$ contains a linear term corresponding to a nonzero mass in the quantum mechanics problem, $W(x) = gx^{1+2\delta} + mx$, these equations and thus the calculation that follows are modified only trivially. Namely, on the left-hand side of Eq. (2.3) each term $gx_i$ is replaced by $(m+g)\, x_i$ and the right-hand sides are unaltered.] We will impose retarded boundary conditions on the Langevin equations:

$$x(t_0) = 0 \tag{2.4}$$

where $t_0$ is the time at which the source $f(t)$ first turns on. Thus $x(t)$ is quiescent before the source term begins to operate. With this choice we can easily integrate the first-order differential equations (2.3) using an integrating factor to obtain:

$$x_0(t) = e^{-gt} \int_{t_0}^{t} dy\, e^{gy} f(y) \tag{2.5a}$$

$$x_1(t) = -ge^{-gt} \int_{t_0}^{t} dy\, e^{gy} x_0(y) \ln x_0^2(y) \tag{2.5b}$$

$$\begin{aligned} x_2(t) = -ge^{-gt} \int_{t_0}^{t} dy\, e^{gy} \{&2x_1(y) + x_1(y) \ln x_0^2(y) \\ &+ \tfrac{1}{2} x_0(y) [\ln x_0^2(y)]^2\} \end{aligned} \tag{2.5c}$$

We see from the structure of the first three terms that the right-hand side of the equation for $x_n$ depends only on the lower-order results and thus in principle one can calculate all the $x_n$ by iteration.

The next step in the calculation is to determine the stochastic average of a particular correlation function over the white noise. In this paper we focus on the two-point correlation function to order $\delta^2$,

$$\begin{aligned} G_2(\sigma, \tau) &= \langle x(\sigma)\, x(\tau) \rangle \\ &= \langle [x_0(\sigma) + \delta x_1(\sigma) + \delta^2 x_2(\sigma) + \cdots] \\ &\quad \times [x_0(\tau) + \delta x_1(\tau) + \delta^2 x_2(\tau) + \cdots] \rangle \\ &= \langle x_0(\sigma)\, x_0(\tau) \rangle + \delta[\langle x_0(\sigma)\, x_1(\tau) \rangle + \langle x_1(\sigma)\, x_0(\tau) \rangle] \\ &\quad + \delta^2[\langle x_0(\sigma)\, x_2(\tau) \rangle + \langle x_2(\sigma)\, x_0(\tau) \rangle + \langle x_1(\sigma)\, x_1(\tau) \rangle] \\ &\quad + \cdots \end{aligned} \tag{2.6}$$

This is the minimal calculation needed to perform analytic continuation in $\delta$ via diagonal Padé approximants. The calculation of the first two terms of this series (up to order $\delta$) was given in a previous paper,[3] but we will repeat that calculation here for completeness. For simplicity we will choose $t_0 = 0$. Since we are only interested in what happens at large $\tau$ and $\sigma$ with $T = |\tau - \sigma|$ held fixed, this choice of $t_0$ is only a convenience.

In order to perform the stochastic averages over the white noise, one needs a way of interpreting $\ln(x^2)^n$ in Eqs. (2.1) and (2.3). One strategy is, for a given order of $\delta$, to replace the logarithm in Eq. (2.1b) by a set of polynomial interactions that give the same answer to the given order in $\delta$. To order $\delta^2$, Eq. (2.1b) is equivalent to

$$\frac{dx}{dt} + g(\delta + \delta^2)\, x^{2\alpha+1} + g(-\delta + \delta^2)\, x^{2\beta+1} = f \tag{2.7}$$

in the sense that one obtains, instead of (2.3a), (2.3b), and (2.3c),

$$\dot{x}_0 + gx_0 = f \tag{2.8a}$$

$$\dot{x}_1 + gx_1 = -gx_0^{2\alpha+1} + gx_0^{2\beta+1} \tag{2.8b}$$

$$\dot{x}_2 + gx_2 = -gx_0^{2\alpha+1} - gx_0^{2\beta+1} - (2\alpha+1)\, gx_1 x_0^{2\alpha} + (2\beta+1)\, gx_1 x_0^{2\beta} \tag{2.8c}$$

If we solve these equations and apply the operator

$$D = \frac{1}{2}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}\right) + \frac{1}{4}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) \tag{2.9}$$

and set $\alpha = \beta = 0$, we reproduce Eqs. (2.3a), (2.3b), and (2.3c). This method is used in refs. 1 and 2. Alternatively, one can just use the identity

$$(\ln x_0^2)^n = \left(\frac{\partial}{\partial\alpha}\right)^n x_0^{2\alpha}\bigg|_{\alpha=0} \tag{2.10}$$

to replace the logarithms by polynomial vertices. Both methods give the same result. In this paper we will use the latter method.

## 2.1. Zeroth-Order Calculation

Using (1.4) and (2.5a), we have

$$\begin{aligned}\langle x_0(\tau)\, x_0(\sigma)\rangle &= e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle f(t)\, f(s)\rangle \\ &= \frac{e^{-gT}}{g} - \frac{e^{-g(\tau+\sigma)}}{g}\end{aligned} \tag{2.11}$$

At large $\tau$, $\sigma$ we thus obtain the equilibrated result to zeroth order in $\delta$,

$$G_2^{\mathrm{eq}}(\sigma, \tau) = \langle x_0(\tau)\, x_0(\sigma)\rangle_{\mathrm{eq}} = \frac{e^{-gT}}{g} \tag{2.12}$$

where the subscript "eq" denotes the limit $\tau, \sigma \Rightarrow \infty$ with $T = |\tau - \sigma|$ held fixed.

## 2.2. First-Order Calculation

To calculate the contribution to the two-point correlation function to first order in $\delta$, we need to evaluate the two correlation functions $\langle x_0(\sigma)\, x_1(\tau)\rangle$ and $\langle x_1(\sigma)\, x_0(\tau)\rangle$:

$$\langle x_0(\sigma)\, x_1(\tau)\rangle = -g e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle f(s)\, x_0(t) \ln[x_0^2(t)]\rangle \tag{2.13}$$

In order to use (1.4), we rewrite the logarithm using

$$\frac{dx^\alpha}{d\alpha}\bigg|_{\alpha=0} = \ln x \tag{2.14}$$

Thus we obtain

$$\langle x_0(\sigma)\, x_1(\tau)\rangle = -ge^{-g(\sigma+\tau)} \frac{d}{d\alpha} \left\{ \int_0^\sigma ds \int_0^\tau dt\, e^{g(t+s)} e^{-(2\alpha+1)gt} \times \left\langle f(s) \prod_{i=1}^{2\alpha+1} \left[ \int_0^t dz_i\, e^{gz_i} f(z_i) \right] \right\rangle \right\} \Bigg|_{\alpha=0} \tag{2.15}$$

To evaluate the expectation value on the right-hand side of (2.15), notice that, because of the Gaussian statistics of $f$, from the $2\alpha+1$ factors $f(z_i)$ and the one factor $f(s)$ one can make $(2\alpha+1)!!$ different products of pairs of sources. Each leads to the same product of two integrals to be performed. After one of the $f(z_i)$ is paired with $f(s)$ [there are $(2\alpha+1)$ ways to do this], the remaining $2\alpha$ $f(z_i)$ can form $\alpha$ pairs of $f(z_i)$ in $(2\alpha-1)!!$ ways. This is represented by the diagram in Fig. 1. We thus have

$$\left\langle f(s) \prod_{i=1}^{2\alpha+1} \left[ \int_0^t dz_i\, e^{gz_i} f(z_i) \right] \right\rangle = (2\alpha+1)!!\, I_1(t, s)[I_2(t)]^\alpha \tag{2.16}$$

where

$$I_1(t, s) = \int_0^t dz\, e^{gz} \langle f(s)\, f(z)\rangle = 2\Theta(t-s)\, e^{gs} \tag{2.17}$$

and

$$I_2(t) = \int_0^t dz \int_0^t dy\, e^{g(z+y)} \langle f(z)\, f(y)\rangle = \frac{e^{2gt}-1}{g} \tag{2.18}$$

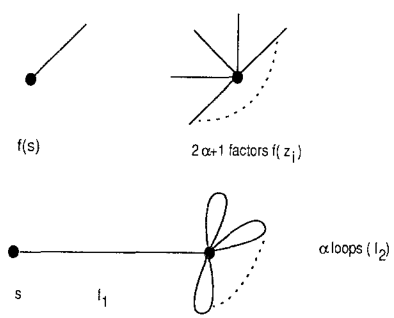


Fig. 1. Graphical representation of (2.16).

We obtain

$$\langle x_0(\sigma)\, x_1(\tau)\rangle = -2ge^{-g(\sigma+\tau)} \frac{d}{d\alpha}\left[(2\alpha+1)!! \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \times e^{-(2\alpha+1)gt}\Theta(t-s)\, e^{gs} \frac{(e^{2gt}-1)^\alpha}{g^\alpha}\right]\Bigg|_{\alpha=0} \tag{2.19}$$

Since we are only interested in the derivative at $\alpha=0$, we can rewrite

$$e^{-2\alpha gt}(e^{2gt}-1)^\alpha = 1-\alpha e^{-2gt}+O(\alpha^2) \tag{2.20}$$

We also notice that the term proportional to $\alpha$ in (2.20), when integrated over $t$, will be exponentially suppressed in the equilibrated regime. Thus,

$$\begin{aligned}\langle x_0(\sigma)\, x_1(\tau)\rangle_{\text{eq}} &= -2ge^{-g(\sigma+\tau)} \frac{d}{d\alpha} \\ &\quad \times\left[(2\alpha+1)!! \int_0^\tau dt \int_0^\sigma ds\, e^{2gs} \frac{\Theta(t-s)}{g^\alpha}\right]\Bigg|_{\alpha=0} \\ &= -(2g)^{-1}\, e^{-g|\sigma-\tau|}[1+2g\Theta(\tau-\sigma)\,|\tau-\sigma|] \\ &\quad \times \frac{d}{d\alpha}\left(\frac{(2\alpha+1)!!}{g^\alpha}\right)\Bigg|_{\alpha=0}\end{aligned} \tag{2.21}$$

Using

$$(2\alpha+1)!! = 2^\alpha \frac{\Gamma(\alpha+3/2)}{\Gamma(3/2)} \tag{2.22}$$

and the Taylor expansion of the $\Gamma$ function for small $\alpha$

$$\Gamma(c+\alpha)=\Gamma(c)[1+\alpha\psi(c)+O(\alpha^2)]$$

we obtain

$$\frac{d}{d\alpha}\left(\frac{(2\alpha+1)!!}{g^\alpha}\right)\Bigg|_{\alpha=0} = L \tag{2.23}$$

and

$$\langle x_0(\sigma)\, x_1(\tau)\rangle_{\text{eq}} = -\frac{L}{2g} e^{-g|\sigma-\tau|}[1+2g\Theta(\tau-\sigma)\,|\tau-\sigma|]$$

where $L=\psi(3/2)+\ln(2/g)$. So

$$\langle x_0(\sigma)\, x_1(\tau)+x_1(\sigma)\, x_0(\tau)\rangle_{\text{eq}} = -\frac{L}{g} e^{-g|\sigma-\tau|}(1+g\,|\tau-\sigma|) \tag{2.24}$$

To order $\delta$ we find[3]

$$G_2^{\mathrm{eq}}(\sigma, \tau) = e^{-gT} \frac{1 - \delta L(1+gT) + \cdots}{g} \tag{2.25}$$

## 2.3. Second-Order Calculation

At order $\delta^2$ we need to calculate both $\langle x_1(\sigma)\, x_1(\tau)\rangle$ and the symmetrized $\langle x_0(\sigma)\, x_2(\tau)\rangle$ in the equilibrated regime. First, consider $\langle x_1(\sigma)\, x_1(\tau)\rangle$:

$$\begin{aligned}
\langle x_1(\sigma)\, x_1(\tau)\rangle &= g^2 e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle x_0(s) \ln x_0^2(s)\, x_0(t) \ln x_0^2(t)\rangle \\
&= \partial_\alpha \partial_\beta\, g^2 e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{-2g(\beta t + \alpha s)} \\
&\quad \times \left\langle \prod_{i=1}^{2\alpha+1} \left[ \int_0^s e^{g z_i} f(z_i)\, dz_i \right] \right. \\
&\quad \times \left. \prod_{j=1}^{2\beta+1} \left[ \int_0^t e^{g y_j} f(y_j)\, dy_j \right] \right\rangle \Bigg|_{\alpha=\beta=0}
\end{aligned} \tag{2.26}$$

In the expectation value on the right-hand side of (2.26) there are $2\alpha+1$ factors $f(z_i)$ and $2\beta+1$ factors $f(y_i)$ corresponding to the lines in Fig. 2a. We denote by $S_k(\alpha, \beta)$ the Gaussian-statistics combinatoric factor corresponding to the contractions of the sources depicted in Fig. 2b, where $2k+1$ factors $f(y_j)$ pair with $2k+1$ factors $f(z_i)$ and the remaining $f(z_i)$

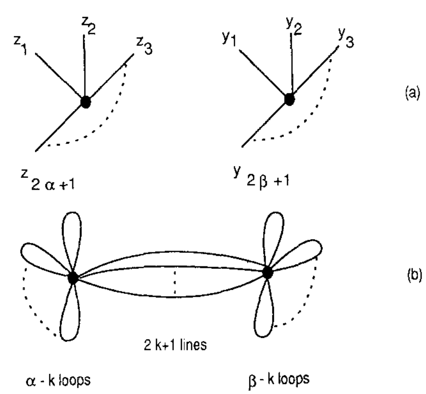


Fig. 2. Graphical representation of (2.27).

and $f(y_j)$ contract in pairs among themselves. We can therefore rewrite the expectation value using the following shorthand:

$$\left\langle \prod_{i=1}^{2\alpha+1} \left[ \int_0^s e^{gz_i} f(z_i)\, dz_i \right] \prod_{j=1}^{2\beta+1} \left[ \int_0^t e^{gy_j} f(y_j)\, dy_j \right] \right\rangle$$

$$= \sum_{k=0}^{\infty} S_k(\alpha, \beta) [I_2(t)]^{\beta-k} [I_3(s, t)]^{2k+1} [I_2(s)]^{\alpha-k} \tag{2.27}$$

where

$$S_k(\alpha, \beta) = \binom{2\alpha+1}{2k+1} \binom{2\beta+1}{2k+1} (2k+1)!\, (2\alpha - 2k - 1)!!\, (2\beta - 2k - 1)!!$$

or

$$S_k(\alpha, \beta) = 2^{\alpha+\beta-2k} \frac{\Gamma(2+2\alpha)\, \Gamma(2+2\beta)\, \Gamma(\alpha - k + 1/2)\, \Gamma(\beta - k + 1/2)}{\Gamma^2(1/2)\, \Gamma(2k+2)\, \Gamma(2\alpha - 2k + 1)\, \Gamma(2\beta - 2k + 1)}$$

$$I_3(s, t) = \int_0^s e^{gz}\, dz \int_0^t e^{gy}\, dy\, \langle f(z)\, f(y) \rangle$$

$$= \Theta(s-t) \frac{e^{2gt} - 1}{g} + \Theta(t-s) \frac{e^{2gs} - 1}{g} \tag{2.28}$$

and $I_2(t)$ is given by (2.18).

Since we are only interested in the equilibrated regime, we can neglect, in both $I_2$ and $I_3$, the 1 in comparison with the exponential. Performing the integrals, we find

$$\langle x_1(\sigma)\, x_1(\tau) \rangle_{\mathrm{eq}} = \partial_\alpha \partial_\beta \sum_{k=0}^{\infty} F_k(|\sigma - \tau|) \left. \frac{S_k(\alpha, \beta)}{g^{\alpha+\beta+1}} \right|_{\alpha=\beta=0} \tag{2.29}$$

where

$$F_k = \frac{2k + (1 - e^{2kg|\sigma+\tau|})}{4k(k+1)} e^{-g|\sigma-\tau|} \tag{2.30}$$

Letting $R_k = \partial_\alpha \partial_\beta S_k(\alpha, \beta)/g^{\alpha+\beta+1}|_{\alpha=\beta=0}$, we find that, for $k = 0$,

$$R_0 = \frac{L^2}{g}; \qquad L = \psi\left(\frac{3}{2}\right) + \ln\left(\frac{2}{g}\right) \tag{2.31}$$

and, for $k \geqslant 1$,

$$R_k = \frac{B(k, 3/2)}{gk}, \qquad \text{where} \quad B\left(k, \frac{3}{2}\right) = \frac{\Gamma(k)\, \Gamma(3/2)}{\Gamma(k+3/2)} \tag{2.32}$$

Thus,

$$\langle x_1(\sigma)\, x_1(\tau)\rangle_{\text{eq}} = \frac{L^2}{2g} e^{-g|\sigma-\tau|}\{1 + g|\sigma-\tau|\}$$
$$+ e^{-g|\sigma-\tau|} \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{2k + (1 - e^{-2kg|\sigma-\tau|})}{4gk^2(k+1)} \tag{2.33}$$

We evaluate the right-hand side of (2.33) using

$$\frac{S_{1,1}(x)}{2g} \equiv \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{2k + (1 - z^k)}{4gk^2(k+1)} \tag{2.34}$$

where $z = e^{-2g|\sigma-\tau|}$. Define

$$\sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{z^k}{k} = F_1(z) \tag{2.35a}$$

$$\sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{z^k}{k+1} = F_2(z) \tag{2.35b}$$

$$\sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{z^k}{k^2} = F_3(z) \tag{2.35c}$$

Then

$$S_{1,1}(z) = \frac{F_1(z) + F_1(1) - F_2(z) - F_2(1) + F_3(1) - F_3(z)}{2} \tag{2.36}$$

The functions $F_i(z)$ are determined in Appendix A. In particular, at equal times ($\sigma = \tau$) we have

$$S_{1,1}(1) = \frac{3\pi^2}{4} - 7$$

Thus

$$\langle x_1(\sigma)\, x_1(\tau)\rangle_{\text{eq}} = (2g)^{-1} e^{-g|\sigma-\tau|}[L^2(1 + g|\sigma-\tau|) + S_{1,1}(e^{-2g|\sigma-\tau|})] \tag{2.37}$$

and

$$\langle x_1(\tau)\rangle^2_{\text{eq}} = \frac{1}{2g}\left(L^2 + \frac{3\pi^2}{4} - 7\right)$$

Next we consider

$$\langle x_0(\tau)\, x_2(\sigma)\rangle = -ge^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)}$$
$$\times \langle f(t)\{2x_1(s) + x_1(s) \ln x_0^2(s) + \tfrac{1}{2} x_0(s)[\ln x_0^2(s)]^2\}\rangle$$
$$\equiv \langle x_0(\tau)\, x_2(\sigma)\rangle_a + \langle x_0(\tau)\, x_2(\sigma)\rangle_b + \langle x_0(\tau)\, x_2(\sigma)\rangle_c \quad (2.38)$$

First consider the "$a$" term in Eq. (2.38):

$$\langle x_0(\tau)\, x_2(\sigma)\rangle_a = -2ge^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle f(t)\, x_1(s)\rangle \quad (2.39)$$

Rewriting the logarithmic terms in $x_1(s)$ using (2.10), we obtain

$$\langle x_0(\tau)\, x_2(\sigma)\rangle_a = 2g^2 e^{-g(\sigma+\tau)} \int_0^\tau dt\, e^{gt} \int_0^\sigma ds$$
$$\times \frac{\partial}{\partial\alpha} \left\{ \int_0^s dr\, e^{-gr\alpha} \left\langle f(t) \prod_{i=1}^{2\alpha+1} \left[ \int_0^r dz_i\, e^{gz_i} f(z_i) \right] \right\rangle \right\} \Bigg|_{\alpha=0} \quad (2.40)$$

Making use of (2.16) and (2.17), we perform the integrals over $z_i$ and $t$ to obtain, in the equilibrated limit,

$$\langle x_0(\tau)\, x_2(\sigma)\rangle_{a,\mathrm{eq}} = 2ge^{-g(\sigma+\tau)} \frac{d}{d\alpha} \left( \frac{(2\alpha+1)!!}{g^\alpha} \right) \Bigg|_{\alpha=0}$$
$$\times \int_0^\sigma ds \int_0^s dr\, [e^{2g\tau} \Theta(r-\tau) + e^{2gr} \Theta(\tau - r)]$$

Performing the remaining two integrals, we have

$$\langle x_0(\tau)\, x_2(\sigma)\rangle_{a;\mathrm{eq}} = \frac{L}{g} e^{-g|\sigma-\tau|} [1 + 2g\Theta(\sigma-\tau)(|\sigma-\tau| + g|\sigma-\tau|^2)] \quad (2.41)$$

so

$$\langle x_0(\tau)\, x_2(\sigma)\rangle + \langle x_0(\sigma)\, x_2(\tau)\rangle_{a,\mathrm{eq}} = \frac{L}{g} e^{-g|\sigma-\tau|} (1 + g|\sigma-\tau| + g^2 |\sigma-\tau|^2) \quad (2.42)$$

and

$$2\langle x_0(\sigma)\, x_2(\sigma)\rangle_{a,\mathrm{eq}} = L/g$$

Next we consider the "$b$" term in Eq. (2.38):

$$\langle x_0(\tau)\, x_2(\sigma)\rangle_b = -g e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle f(t)\, x_1(s) \ln x_0^2(s)\rangle \quad (2.43)$$

Again rewriting the logarithmic term using (2.10), we have

$$\begin{aligned}\langle x_0(\tau)\, x_2(\sigma)\rangle_b = {} & g^2 e^{-g(\sigma+\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{gt} \int_0^s dr \\ & \times \partial_\beta \partial_\alpha \left\{ e^{-2g(s\alpha + r\beta)} \left\langle f(t) \prod_{i=1}^{2\alpha} \left[ \int_0^s dz_i\, e^{gz_i} f(z_i)\right] \right.\right. \\ & \left.\left.\left. \times \prod_{j=1}^{2\beta+1} \left[ \int_0^r dy_j\, e^{gy_j} f(y_j)\right] \right\rangle \right\}\right|_{\alpha=\beta=0} \end{aligned} \quad (2.44)$$

Taking the expectation value and using the fact that $f$ obeys Gaussian statistics leads to two different types of terms. First, referring to Fig. 3a, $f(t)$ can be contracted with any one of the $f(z_i)$, leaving $2\alpha - 1$ factors $f(z_i)$. Then, $2k+1$ pairwise contractions of the form $\langle f(y_j)\, f(z_i)\rangle$ are done,

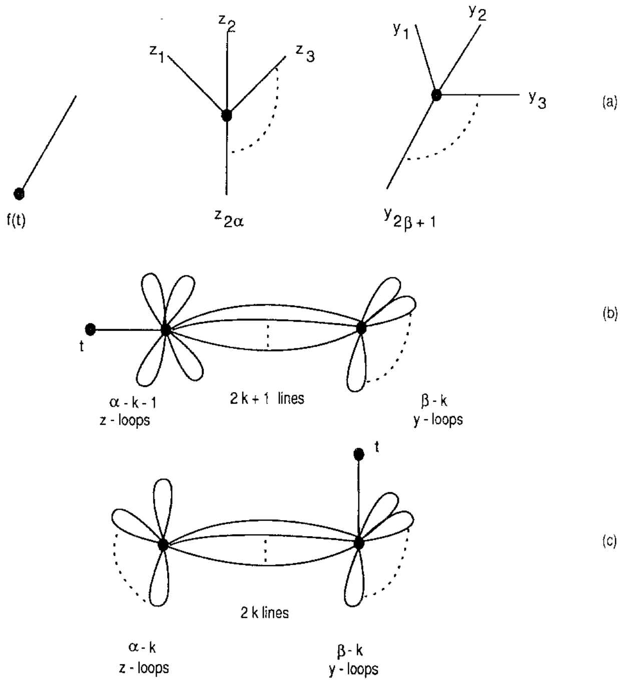


Fig. 3. Graphical representation of (2.47).

leaving $2\alpha - 2k - 2$ pairwise contractions of the remaining $f(z_i)$ and $2\beta - 2k$ pairwise contractions of the $f(y_i)$. This type of term is illustrated in Fig. 3b and has a statistical weight given by $F_{1k}(\alpha, \beta)$, where

$$F_{1k}(\alpha, \beta) = 2\alpha \binom{2\alpha - 1}{2k+1} (2k+1)! \binom{2\beta + 1}{2k+1} (2\alpha - 2k - 3)!! \, (2\beta - 2k - 1)!! \tag{2.45}$$

Alternatively, $f(t)$ can be contracted with one of the $f(y_i)$, leaving $2\beta$ factors $f(y_i)$. Then $2k$ pairwise contractions of the form $\langle f(y_i) f(z_i) \rangle$ are done, leaving $2\alpha - 2k$ pairwise contractions of the $f(z_i)$ and $2\beta - 2k$ pairwise contractions of the $f(y_i)$. This type of term is illustrated in Fig. 3c and has a statistical weight given by $F_{2k}(\alpha, \beta)$, where

$$F_{2k}(\alpha, \beta) = (2\beta + 1) \binom{2\beta}{2k} (2k)! \binom{2\alpha}{2k} (2\alpha - 2k - 1)!! \, (2\beta - 2k - 1)!! \tag{2.46}$$

We therefore can write

$$\begin{aligned} &\int_0^\tau dt\, e^{gt} \left\langle f(t) \prod_{i=1}^{2\alpha} \left[ \int_0^s dz_i\, e^{gz_i} f(z_i) \right] \prod_{j=1}^{2\beta+1} \left[ \int_0^r dy_j\, e^{gy_j} f(y_j) \right] \right\rangle \\ &\quad = \sum_{k=0}^{\infty} F_{1k}(\alpha, \beta)\, I_3(\tau, s) [I_2(r)]^{\beta - k} [I_3(r, s)]^{2k+1} [I_2(s)]^{\alpha - k - 1} \\ &\qquad + \sum_{k=0}^{\infty} F_{2k}(\alpha, \beta)\, I_3(\tau, r) [I_2(r)]^{\beta - k} [I_3(r, s)]^{2k} [I_2(s)]^{\alpha - k} \end{aligned} \tag{2.47}$$

where $I_2$ and $I_3$ are defined in (2.18) and (2.28). Using the leading contribution to $I_2$ and $I_3$ relevant in the equilibrated limit, we obtain

$$\begin{aligned} \langle x_0(\tau)\, x_2(\sigma) \rangle_{b,\mathrm{eq}} = g e^{-g(\sigma + \tau)} \int_0^\sigma ds \int_0^s dr \\ \times \Bigg\{ \sum_{k=0}^{\infty} R_{1k} [\Theta(\tau - s)\, e^{-2gsk} e^{2gr(k+1)} \\ + \Theta(s - \tau)\, e^{2g\tau} e^{-2gs(k+1)} e^{2gr(k+1)}] \\ + \sum_{k=0}^{\infty} R_{2k} [\Theta(\tau - r)\, e^{-2gsk} e^{2gr(k+1)} \\ + \Theta(r - \tau)\, e^{2g\tau} e^{-2gsk} e^{2grk}] \Bigg\} \end{aligned}$$

where

$$R_{ik} = \partial_\alpha \partial_\beta \left. \frac{F_{ik}(\alpha, \beta)}{g^{\alpha+\beta}} \right|_{\alpha=\beta=0} \tag{2.48}$$

Performing the integrals, we obtain

$$\begin{aligned}
&\langle x_0(\tau)\, x_2(\sigma)\rangle_{\text{eq},b} \\
&\quad = \frac{1}{4g} e^{-g|\sigma-\tau|}[1 + 2g\,|\sigma-\tau|\,\Theta(\sigma-\tau)] \left( \sum_{k=1}^{\infty} \frac{R_{1k}}{k+1} + R_{1,k=0} \right) \\
&\quad\quad + \frac{1}{4g} e^{-g|\sigma-\tau|}\{1 + \Theta(\sigma-\tau)[2g\,|\sigma-\tau| + 2g^2\,|\sigma-\tau|^2]\}\, R_{2,k=0} \\
&\quad\quad + \frac{1}{4g} e^{-g|\sigma-\tau|} \sum_{k=1}^{\infty} R_{2k} \left[ \frac{1}{k+1} + \Theta(\sigma-\tau) \right. \\
&\quad\quad \left. \times \left( \frac{2g}{k}\,|\sigma-\tau| + \frac{e^{-2gk|\sigma-\tau|} - 1}{k^2(k+1)} \right) \right]
\end{aligned}$$

Symmetrizing with respect to $\sigma$ and $\tau$ yields

$$\begin{aligned}
&\langle x_0(\tau)\, x_2(\sigma) + x_0(\sigma)\, x_2(\tau)\rangle_{b,\text{eq}} \\
&\quad = \frac{1}{2g} e^{-g|\sigma-\tau|}(1 + g\,|\sigma-\tau|) \left( \sum_{k=1}^{\infty} \frac{R_{1k}}{k+1} + R_{1,k=0} \right) \\
&\quad\quad + \frac{1}{2g} e^{-g|\sigma-\tau|}(1 + g\,|\sigma-\tau| + g^2\,|\sigma-\tau|^2)\, R_{2,k=0} \\
&\quad\quad + \frac{1}{2g} e^{-g|\sigma-\tau|} \sum_{k=1}^{\infty} R_{2k} \left( \frac{1}{k+1} + \frac{g\,|\sigma-\tau|}{k} + \frac{e^{-2gk|\sigma-\tau|} - 1}{2k^2(k+1)} \right)
\end{aligned} \tag{2.49}$$

Using (2.45), (2.46), and (2.48),

$$R_{1,k=0} = 2L \tag{2.50a}$$

$$R_{2,k=0} = L^2 - 2L \tag{2.50b}$$

and, for $k \geqslant 1$,

$$R_{1k} = -2^{2k+1} \frac{B(1+k, 1+k)}{k} = -2B\left(k, \frac{3}{2}\right) \tag{2.50c}$$

$$R_{2k} = \frac{4^k(1+2k)\, B(1+k, 1+k)}{k^2} = \frac{4^k B(k, k)}{2k} = \frac{B(k, 1/2)}{k} \tag{2.50d}$$

To do the sums in (2.49), it is useful to define the functions

$$\begin{aligned} H_0(z) &= \sum_{k=1}^{\infty} B(k,k)\, z^k, \qquad H_1(z) = \sum_{k=1}^{\infty} B(k,k)\frac{z^k}{k} \\ H_2(z) &= \sum_{k=1}^{\infty} B(k,k)\frac{z^k}{k+1}, \qquad H_3(z) = \sum_{k=1}^{\infty} B(k,k)\frac{z^k}{k^2} \\ H_4(z) &= \sum_{k=1}^{\infty} B(k,k)\frac{z^k}{k^3} \\ A &= \sum_{k=1}^{\infty} \frac{R_{1k}}{k+1} = -2\sum_{k=1}^{\infty} 4^k \frac{B(1+k,1+k)}{k(k+1)} = \frac{\pi^2}{2} - 6 \end{aligned} \tag{2.51}$$

In Appendix A we perform the sums over $k$ in $A$ and in the $H_i(z)$. In terms of the $H_i(z)$,

$$\sum_{k=1}^{\infty} \frac{R_{2k}}{k+1} = \frac{H_1(4)}{2} - \frac{H_2(4)}{2} = \frac{\pi^2}{4} - 1$$

$$\sum_{k=1}^{\infty} \frac{R_{2k}}{k} = \frac{H_3(4)}{2} = 2.63389...$$

$$\begin{aligned} \sum_{k=1}^{\infty} R_{2k} \frac{e^{-2gk\,|\sigma-\tau|} - 1}{2k^2(k+1)} &\equiv S_{0,2}(e^{-2g\,|\sigma-\tau|}) \\ &= \frac{1}{4}\,[H_4(4e^{-2g\,|\sigma-\tau|}) - H_4(4) - H_3(4e^{-2g\,|\sigma-\tau|}) \\ &\quad + H_3(4) + H_1(4e^{-2g\,|\sigma-\tau|}) - H_1(4) \\ &\quad - H_2(4e^{-2g\,|\sigma-\tau|}) + H_2(4)] \\ S_{0,2}(1) &= 0 \end{aligned} \tag{2.52}$$

Then Eq. (2.49) gives

$$\begin{aligned} &\langle x_0(\tau)\, x_2(\sigma) + x_0(\sigma)\, x_2(\tau)\rangle_{b,\mathrm{eq}} \\ &\quad = (2g)^{-1}\, e^{-g\,|\sigma-\tau|} \left\{ \frac{3\pi^2}{4} - 7 + L^2 + g\,|\sigma-\tau| \left[ \frac{\pi^2}{2} - 6 + L^2 + \frac{H_3(4)}{2} \right] \right. \\ &\qquad \left. + g^2\,|\sigma-\tau|^2\,(L^2 - 2L) + S_{0,2}(e^{-2g\,|\sigma-\tau|}) \right\} \end{aligned} \tag{2.53a}$$

$$2\langle x_0(\tau)\, x_2(\tau)\rangle = \frac{1}{2g}\left( \frac{3\pi^2}{4} - 7 + L^2 \right) \tag{2.53b}$$

Next we calculate the term $\langle x_0(\sigma)\, x_2(\tau)\rangle_{c,\mathrm{eq}}$ in Eq. (2.38):

$$\langle x_0(\sigma)\, x_2(\tau)\rangle_c = -\frac{1}{2} e^{-g(\sigma-\tau)} \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} \langle f(s)\, x_0(t) \ln^2[x_0^2(t)]\rangle \tag{2.54}$$

Using (2.10),

$$\langle x_0(\sigma)\, x_2(\tau)\rangle_c = -\frac{1}{2} e^{-g(\sigma-\tau)} \frac{d^2}{d\alpha^2} \left\{ \int_0^\tau dt \int_0^\sigma ds\, e^{g(t+s)} e^{-(2\alpha+1)gt} \times \left\langle f(s) \prod_{i=1}^{2\alpha+1} \left[ \int_0^t dz_i\, e^{gz_i} f(z_i) \right] \right\rangle \right\} \Bigg|_{\alpha=0} \tag{2.55}$$

The integrals above were performed earlier in the paper [see (2.15)ff.] so

$$\begin{aligned}\langle x_0(\sigma)\, x_2(\tau)\rangle_{c,\mathrm{eq}} &= -\frac{1}{4g}\, g e^{-g|\sigma-\tau|}[1+2g\Theta(\tau-\sigma)\,|\tau-\sigma|] \times \frac{d^2}{d\alpha^2}\left[\left(\frac{2}{g}\right)^\alpha \frac{\Gamma(\alpha+3/2)}{\Gamma(3/2)}\right]\Bigg|_{\alpha=0} \\ &= -\frac{1}{4g} e^{-g|\sigma-\tau|}[1+2g\Theta(\tau-\sigma)\,|\tau-\sigma|]\left[L^2+\psi'\left(\frac{3}{2}\right)\right]\end{aligned} \tag{2.56}$$

where $\psi'(3/2)=\pi^2/(2)-4$. Symmetrizing (2.56) gives

$$\langle x_0(\sigma)\, x_2(\tau) + x_0(\tau)\, x_2(\sigma)\rangle_{c,\mathrm{eq}} = -\frac{1}{2g} e^{-g|\sigma-\tau|}(1+g\,|\tau-\sigma|)\left[L^2+\psi'\left(\frac{3}{2}\right)\right] \tag{2.57}$$

Combining the three terms in Eqs. (2.38), (2.42), (2.49), and (2.57), we obtain the equilibrated two-point correlation function to order $\delta^2$,

$$\begin{aligned}G_2^{\mathrm{eq}}(\sigma,\tau) = \frac{1}{2g} e^{-g|\sigma-\tau|}\Bigg\{ & 2-2\delta L(1+g\,|\sigma-\tau|) \\ & +\delta\Bigg[L^2(1+g\,|\sigma-\tau|+g^2\,|\sigma-\tau|^2) \\ & +(2L-2)(1+g\,|\sigma-\tau|) \\ & +\left(\frac{\pi^2}{4}-1\right)+g\frac{|\sigma-\tau|}{2}H_3(4)+S_{11}(z)+S_{02}(z)\Bigg]\Bigg\}\end{aligned} \tag{2.58}$$

where $z = e^{-2g|\sigma - \tau|}$. When

$$\sigma = \tau, \qquad S_{11}(1) = \frac{3\pi^2}{4} - 7, \qquad S_{02}(1) = 0$$

and

$$G_2^{\text{eq}}(\sigma\sigma) = \frac{1}{g}\left\{1 - \delta L + \delta^2\left(\frac{L^2}{2} + L + \psi'\left(\frac{3}{2}\right) - 1\right)\right\} \tag{2.59}$$

## 3. ANALYTIC CONTINUATION IN δ

In order to explore the nonlinear regime, one needs to analytically continue the $\delta$ series outside its radius of convergence. One effective way of doing this is by means of Padé approximants.[4] The equal-time correlation functions are known analytically for all $\delta$ from (1.12) and for these one can exactly test the domain of validity of the Padé approximants. For the unequal-time correlation functions one has to rely on a comparison with the numerical solution of the Langevin equation.

First consider the equal-time correlation functions. From the time-independent solution of the Fokker–Planck equation (1.10) we have

$$\langle x^n \rangle_{\text{eq}} = \frac{\int_{-\infty}^{\infty} dx\, x^n \exp[-gx^{2(1+\delta)}/(2+2\delta)]}{\int_{-\infty}^{\infty} dx \exp[-gx^{2(1+\delta)}/(2+2\delta)]} \tag{3.1}$$

Thus the equal-time correlation function, exact to all orders in $\delta$, is

$$G_2(t, t) = \left(\frac{2+2\delta}{g}\right)^{1/(1+\delta)} \frac{\Gamma(3/(2+2\delta))}{\Gamma(1/(2+2\delta))} \tag{3.2}$$

The first three terms in the Taylor series in $\delta$ of Eq. (3.2) are

$$G_2(t, t) = \frac{1}{g}\left\{1 - \delta L + \delta^2\left[-1 + L + \frac{1}{2}L^2 + \psi'\left(\frac{3}{2}\right)\right] + \cdots\right\} \tag{3.3}$$

where $L = \ln(2/g) + \psi(3/2)$.

This agrees precisely with Eq. (2.59), obtained by solving the Langevin equation in the $\delta$ expansion.

Note that Eq. (3.2) has an essential singularity at $\delta = 1$, so the Taylor series (3.3) has radius of convergence 1.

To analytically continue the approximate result (2.59) to large $\delta$, we form the [1, 1] Padé approximant, which agrees with this Taylor series up to $\delta^2$:

$$G_2^{[1,1]}(t, t) = g^{-1}\,\frac{2L + (2\psi'(3/2) - L^2 + 2L - 2)\delta}{2L + (2\psi'(3/2) + L^2 + 2L - 2)\delta} \tag{3.4}$$

One immediate drawback of this Padé approximant is that at small $g$ and at large $g$, Eq. (3.4) behaves like $1/g$, whereas the exact answer, Eq. (3.2), behaves like $(1/g)^{1/(1+\delta)}$. In Section 4 we will show how to remedy this problem. In Fig. 4 we compare the exact answer at $g=1$, as a function of $\delta$, with the [1, 1] Padé. We notice the excellent agreement from $\delta=0$ to $\delta=5$. (The error at $\delta=5$ is approximately 6%, even though this point is far outside the radius of convergence of the $\delta$ expansion.)

In Fig. 5 we compare the exact answer for $\delta=1$ as a function of $g$ with the [1, 1] Padé. Although the square-root singularity at $g=0$ is not correctly obtained, for $g>1/2$ the agreement is excellent.

At $g=2$ the term $\ln(2/g)$ vanishes. Very near this point the numerator and denominator of the Padé approximant (3.4) each have zeros. The failure of these zeros to coincide produces the rapid oscillation near $g=2$ shown in Fig. 5.

If we had included a linear term in $W(x)$, i.e., a mass term in the quantum mechanics problem, so that $W(x)=mx+gx^{\delta+1}$, then the calculation of the $\delta$ expansion would have been almost identical to the preceding one,

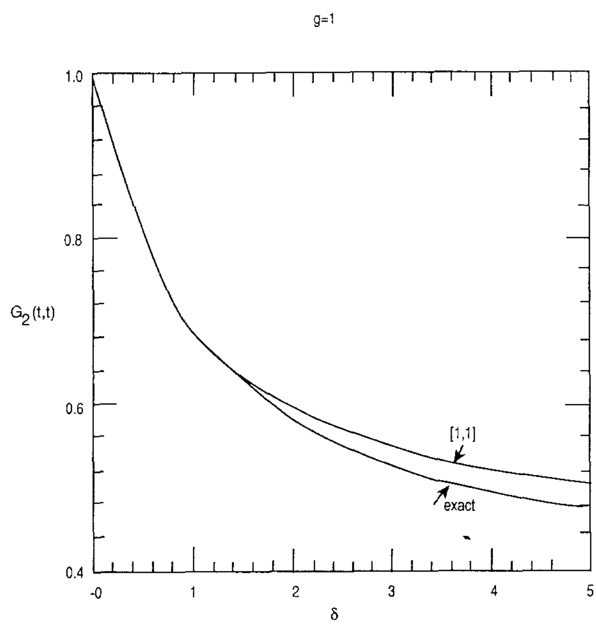


Fig. 4. Comparison, as a function of $\delta$ at fixed $g=1$, of the exact equal-time correlation function given by (3.2) with the [1, 1] Padé approximant (3.4), obtained from the $\delta$ expansion through order $\delta^2$.

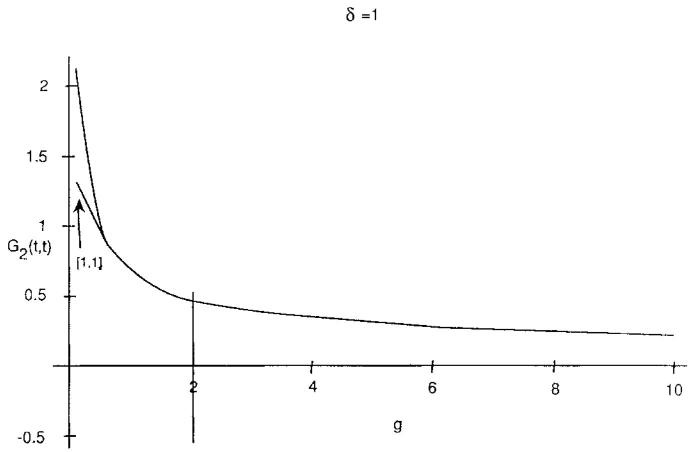


Fig. 5. Comparison of (3.2) and (3.4) for fixed $\delta = 1$ as a function of $g$.

but with $g$ replaced by $m+g$ as discussed below (2.3). This would have allowed a weak coupling expansion in $g$. That is, for $m$ not zero,

$$\langle x^2 \rangle_{\text{eq}} = \frac{\int_{-\infty}^{\infty} dx\, x^2 \tilde{P}(x)}{\int_{-\infty}^{\infty} dx\, \tilde{P}(x)}$$

where

$$\tilde{P}(x) = \exp\left[-\left(\frac{mx^2}{2} + \frac{gx^{2(1+\delta)}}{2+2\delta}\right)\right] \tag{3.5}$$

The $\delta$ expansion of $\tilde{P}(x)$ is

$$\tilde{P}(x) = e^{-(m+g)x^2/2}\left\{1 - \frac{\delta}{2}(2gx^2 \ln x - gx^2)\right. \tag{3.6}$$

$$\left. + \frac{\delta^2}{8}[(4g^2x^4 - 8gx^2)\ln^2 x + (8gx^2 - 4g^2x^4)\ln x + g^2x^4 - 4gx^2] + \cdots\right\}$$

The integrals involved are all straightforward and can be summarized by

$$J(n) \equiv \int_0^{\infty} dx\, x^{2n} e^{-(m+g)x^2/2} = \left(\frac{2}{m+g}\right)^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)} \tag{3.7a}$$

$$\int_0^{\infty} dx \ln x x^{2n} e^{-(m+g)x^2/2} = \frac{1}{2} J(n)\, L(n) \tag{3.7b}$$

where $L(n) = \psi(n+1/2) + \ln[2/(m+g)]$; and

$$\int_0^\infty dx \ln^2 x x^{2n} e^{-(m+g)x^2/2} = \frac{1}{4} J(n)\, L_2(n) \tag{3.7c}$$

where $L_2(n) = \psi'(n+1/2) + L^2(n)$.

To improve the accuracy of the $\delta$ expansion, we again use Padé approximants. To avoid the kind of oscillation seen in Fig. 5, in this case at $\ln[2/(m+g)] = 0$, we use (3.6) to obtain separate $\delta$ expansions of the numerator and denominator of (3.5). We then take the ratio of the [1, 1] Padé approximants. The result, shown in Fig. 6, is accurate to 15% for all $g \in [0, 5]$.

In Appendix B we discuss how well weak and strong coupling expansions work with and without Padé approximants.

Next, let us turn our attention to the unequal-time correlation function given in Eq. (2.58). To evaluate this, we approximate $S_{11}(z)$ and $S_{02}(z)$ by summing one million terms (10,000 would be sufficient) in the series representation for these functions given in Eqs. (2.34) and (2.52) and making a table of values for the interval [0, 1]. We then compare the [1, 1] Padé approximant obtained from Eq. (2.58) as well as the naive result of the order $\delta$ and order $\delta^2$ calculation with a numerical simulation of the Langevin equation (1.2) where the stirring forces obey Gaussian statistics described by (1.3). For the values $g = 1$ and $\delta = 1$ we plot the results of this comparison in Fig. 7. The two lines for the Langevin simulation reflect one standard deviation about the average taken over a large number of independent simulations. We notice that at $\delta = 1$, which is the

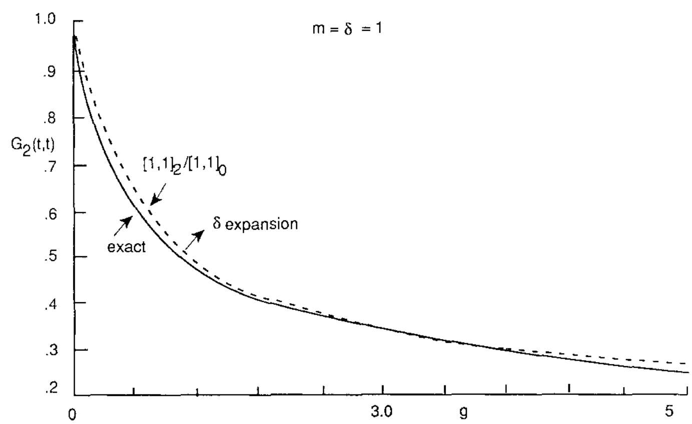


Fig. 6. Comparison of the exact result for $\langle x^2 \rangle$ from (3.5) for $m = \delta = 1$ with the approximate result obtained as a ratio of [1, 1] Padé approximants.

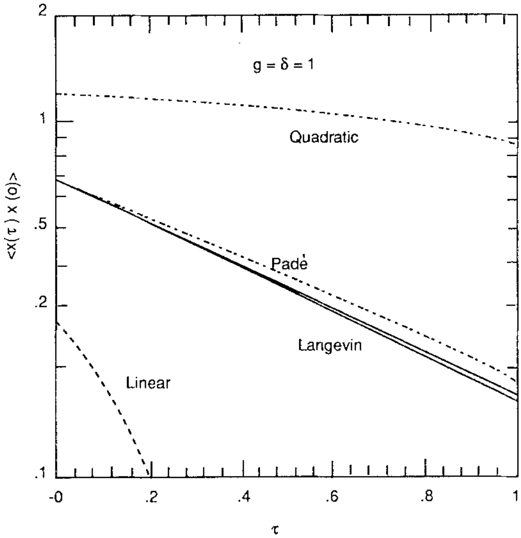


Fig. 7. Evaluation of the correlation function $\langle x(\tau)\, x(0)\rangle$ for $g=\delta=1$. Langevin denotes the numerical solution including error bars. Padé denotes the [1, 1] Padé of the $\delta$ expansion equation (2.58) up to order $\delta^2$.

radius of convergence of the delta expansion for $\tau=0$, the linear and quadratic approximations are quite poor, but the [1, 1] Padé approximant obtained from the quadratic approximation gives excellent agreement over the entire range of $\tau$ plotted. Thus, the analytically continued $\delta$ expansion is quite accurate after only calculating to order $\delta^2$.

## 4. IMPROVING THE ANALYTIC STRUCTURE IN *g* BY USING SCALING LAWS

Notice that the expansion (3.3) has an analytic structure in $g$ that is totally different from the exact expression (3.2). Except at $\delta=0$, (3.2) has a branch point at $g=0$ associated with the power $1/(1+\delta)$. In contrast, (3.3) has a single pole at $g=0$ plus logarithmic branch points. The Padé approximants cannot remedy this. On the other hand, as known from mean field theory, a self-consistent answer does reproduce the correct analytic structure at $g=0$. So we might ask, is there a way of introducing a new parameter into the theory so as to force the correct analytic structure at $g=0$? We do this by exploiting the freedom to make engineering scaling changes on the units used to measure the variable $x$. A complete discussion of this idea will be published elsewhere. Here we will confine ourselves to the explicit calculation of the equal-time correlation function $G_2$.

To force the series (3.3) in $\delta$ to have the correct analytic structure in $g$, one introduces two new parameters $M$ and $b$ as follows:

$$\langle x^2 \rangle = \frac{\int x^2 \exp[-gM^b(x/M)^{2+2\delta}/(2+2\delta)]\,dx}{\int \exp[-gM^b(x/M)^{2+2\delta}/(2+2\delta)]\,dx} \tag{4.1}$$

The exact value of $\langle x^2 \rangle$ is

$$\langle x^2 \rangle = M^2 \left(\frac{2+2\delta}{gM^b}\right)^{2/(2+\delta)} \frac{\Gamma(3/(2+2\delta))}{\Gamma(1/(2+2\delta))} \tag{4.2}$$

We notice that when $b = 2+2\delta$, $\langle x^2 \rangle$ is independent of $M$. (Also note that $g$ now has different dimensions than before.) Thus,

$$\left.\frac{d\langle x^2 \rangle}{dM}\right|_{b=2+2\delta} = 0 \tag{4.3}$$

We realize that if we let $x \Rightarrow \lambda x$ in Eq. (4.1), then dimensional analysis implies

$$\langle x^2 \rangle(\lambda^{d_g} g, \lambda^s M) = \lambda^2 \langle x^2 \rangle(g, M) \tag{4.4}$$

where the dimension of the parameter $M$ is $s$ and the dimension of $g$ is thus [from Eq. (4.1)] $d_g = -sb + (s-1)(2+2\delta)$. Differentiating with respect to $\lambda$ and then setting $\lambda = 1$, we obtain

$$\left(sM \frac{\partial}{\partial M} + d_g\, g \frac{\partial}{\partial g} - 2\right) \langle x^2 \rangle(g, M) = 0 \tag{4.5}$$

This equation should not depend on the dimension of $M$. Differentiating with respect to $s$, one obtains

$$\left[M \frac{\partial}{\partial M} + (2+2\delta-b)\, g \frac{\partial}{\partial g}\right] \langle x^2 \rangle(g, M) = 0 \tag{4.6}$$

which implies $M\,\partial/\partial M = 0$ at $b = 2+2\delta$, which is (4.3).

We can write (4.2) as

$$\langle x^2 \rangle = M^2 f(y, \delta) \tag{4.7}$$

where the generic structure of $f$ is

$$f(y, \delta) = y^{h(\delta)} g(\delta) = e^{h(\delta)\ln y} g(\delta) \tag{4.8}$$

and

$$y = gM^b, \qquad h = -1/(\delta+1) \tag{4.9}$$

The scaling condition (4.3) can be written as

$$[2f(y) + ybf'(y)]\,|_{b=2+2\delta} = 0 \tag{4.10}$$

This equation is an identity if we use the exact equation for $f(y)$ or its expansion to all orders in $\delta$. However, if we calculate only to order $\delta^N$, we have

$$f_N(y) = \sum_{n=0}^{N} a_n(y)\,\delta^n/n! \tag{4.11}$$

and $f_N(y)$ is not independent of $M$.

The scaling condition then leads to the relationship

$$b_N(y) = da_N(y)/dy = 0 \tag{4.12}$$

which allows us to choose a particular $M$ so that (4.3) can be satisfied.

Clearly $b_N(y)$ is a polynomial of degree $N$ in $\ln(y)$. Equation (4.12) has $N$ real roots, but only the smallest root is related to the $N=1$ solution. We denote this root by $y_N^*$.

The sequence of $y_N^*$, for $N = 1, 2,..., 8$, is

$$\begin{aligned}&5.63861, 3.56103, 2.95104, 2.66687,\\ &\quad 2.50419, 2.39919, 2.32591, 2.28717,...\end{aligned} \tag{4.13}$$

This sequence of numbers can be fit by

$$1.8916 + 2.9086/N + 0.839096/N^2 \tag{4.14}$$

Now, $y_N^* = gM^b$. Solving for $M$ as a function of $y^*(N)$ and $b$, we obtain for the $N$th-order improved calculation:

$$\langle x^2 \rangle_N = \left[\frac{y_N^*}{g}\right]^{2/b} f_N(y_N^*) \tag{4.15}$$

where $f$ is defined in (4.11).

We notice that if we now set $b = 2 + 2\delta$, we automatically get the correct analytic behavior as a function of $g$. The next thing to note is that

$$\frac{\Gamma(3/(2+2\delta))}{\Gamma(1/(2+2\delta))}$$

is finite as $\delta \to \infty$. Thus, in trying to extrapolate to large $\delta$ using Padé approximants, one should use diagonal Padé approximants which also have the property of beging finite as $\delta \to \infty$. So the strategy is to calculate to order $\delta^N$ and to solve the scaling condition for $y_N^*$. One then forms the $[N/2, N/2]$ Padé approximant of this answer. Only at the end does one set $b = 2\delta + 2$.

To order $\delta$ we have

$$y_1^* = \tfrac{1}{2} \exp[-(\gamma_E - 3)] \tag{4.16}$$

where $\gamma_E$ is Euler's constant, $\gamma_E \sim 0.57...$ .

Inserting this in the expansion of the answer up to order $\delta$, one obtains as the lowest order result

$$\langle x^2 \rangle_{N=1} = (2\delta + 2)\{(2g)^{-1} \exp[-(\gamma_E - 3)]\}^{1/(\delta+1)} \tag{4.17}$$

At $\delta = 1$ this yields

$$\left(\frac{0.842255}{g}\right)^{1/2}$$

as opposed to the exact answer

$$\left(\frac{0.675977}{g}\right)^{1/2}$$

As expected, this approximation has the correct analytic behavior in $g$ and the coeficient is accurate to 10%.

To order $\delta^2$, we obtain instead

$$y_2^* = \tfrac{1}{2} \exp[4 - \gamma_E - (12 - \pi^2)^{1/2}] \tag{4.18}$$

Inserting this value of $y^*$ into the $[1, 1]$ Padé approximant of the Taylor series in $\delta$ of $\langle x^2 \rangle$, we obtain

$$\frac{\exp\{[\delta/(\delta+1)](\gamma_E - 4 + \sqrt{12 - \pi^2})\}[4 - 2\sqrt{12 - \pi^2} + \delta(30 - 6\sqrt{12 - \pi^2})]}{(2g)^{1/(\delta+1)}[(\sqrt{12 - \pi^2} - 1)\delta + 2 - \sqrt{12 - \pi^2}]} \tag{4.19}$$

Again we obtain the correct analytic behavior in $g$ and the ratio of (4.19) to the exact answer (4.2) is a monotonic function of $\delta$, denoted by $R_1$, having a value 1 at $\delta = 0$ and 1.37779 as $\delta \to \infty$. At $\delta = 1$ the ratio is 1.0128..., showing a great improvement over just doing the calculation up to order $\delta$. The ratio of the improved $[1, 1]$ Padé as compared to the exact answer for all $\delta$ is shown in Fig. 8. As one keeps higher and higher terms in the $\delta$ expansion the ratio of the improved Padé approximants to the exact answer gets better and better.

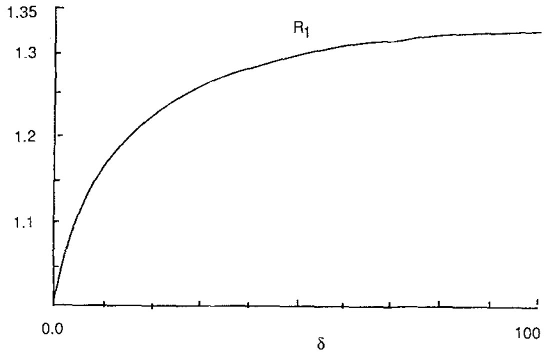


Fig. 8. The ratio $R_1$ of the scaling-improved [1, 1] Padé approximant to the exact answer for $\langle x^2 \rangle$, as a function of $\delta$.

In general, in order to obtain the scaling-relation improved approximation to the answer, we first keep $N$ terms in the $\delta$ expansion (where $N$ is even):

$$\langle x^2 \rangle_N = M^2 \sum_{n=0}^{N} a_n(y)\, \delta^n \tag{4.20}$$

We next form the $[N/2, N/2]$ Padé approximant for that Taylor series:

$$M^2 P[N/2, N/2](y) = M^2 \frac{\sum_{n=0}^{N/2} b_n(y)\, \delta^n}{\sum_{n=0}^{N/2} c_n(y)\, \delta^n} \tag{4.21}$$

We then evaluate this at $y = y_N^* = gM^{2+2\delta}$ to obtain

$$\langle x^2 \rangle_{N, y_N^*} = (y_N^*/g)^{1/(1+\delta)}\, P[N/2, N/2](y_N^*) \tag{4.22}$$

So that we can compare this answer to the exact answer, we form the ratio

$$R_{N/2}(\delta) = \frac{\langle x^2 \rangle_{N, y_N^*}}{\langle x^2 \rangle} \tag{4.23}$$

We plot in Fig. 9 the ratios $R_2$ and $R_3$. By the time we get to $R_3$ the ratio is only 6% high at $\delta = \infty$. At $\delta = 1$ we obtain

$$R_2(\delta = 1) = 1.000170696 \tag{4.24}$$

$$R_3(\delta = 1) = 1.0000039263 \tag{4.25}$$

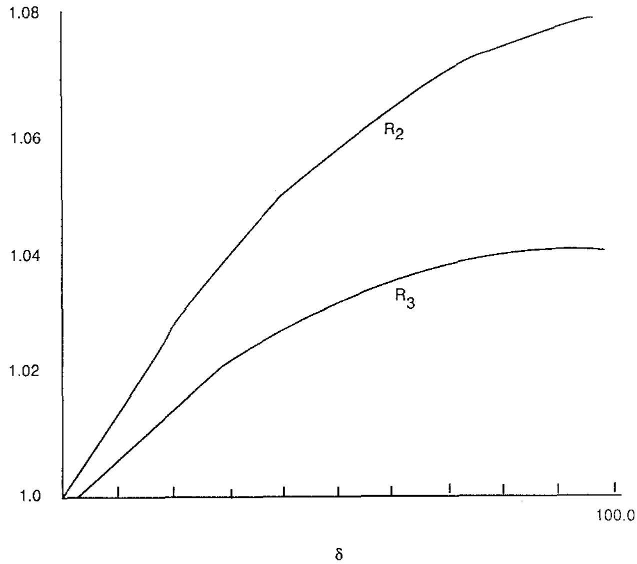


Fig. 9. The ratios $R_2$ and $R_3$ of the scaling-improved [2, 2] and [3, 3] Padé approximants to the exact answer for $\langle x^2 \rangle$, as a function of $\delta$.

Thus, we see that the naive $\delta$ expansion series, when improved by using Padé approximants which are evaluated at $y^*$, the value given by the scaling relation, gives an approximation to the exact answer that has the exact functional dependence on the coupling constant $g$ and gives a uniform approximation for all $\delta$ when $N$ is sufficiently large.

## APPENDIX A. SUMMING THE SERIES

Consider

$$S_{1,1} = \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{2k + (1 - e^{-2kg|\sigma - \tau|})}{2k^2(k+1)} \tag{A1}$$

If we let

$$\begin{gathered} \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{x^k}{k} = F_1(x) \\ \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{x^k}{k+1} = F_2(x) \\ \sum_{k=1}^{\infty} B\left(k, \frac{3}{2}\right) \frac{x^k}{k^2} = F_3(x) \end{gathered} \tag{A2}$$

then we can write

$$S_{1,1}=\frac{F_1(x)+F_1(1)-F_2(x)-F_2(1)+F_3(1)-F_3(x)}{2} \tag{A3}$$

Using the integral representation for the beta function, we have

$$\begin{aligned}\frac{x\,dF_1}{dx}&=\sum_{k=1}^{\infty} B\left(k,\frac{3}{2}\right)x^k\\ &=\sum_{k=1}^{\infty} x^k\int_0^1 dt\,t^{k-1}(1-t)^{1/2}\\ &=x\int_0^1 dt\,(1-xt)^{-1}\,(1-t)^{1/2}\\ &=2-2\left(\frac{1-x}{x}\right)^{1/2}\arctan\left[\left(\frac{x}{1-x}\right)^{1/2}\right]\end{aligned} \tag{A4}$$

Thus we obtain

$$\begin{gathered}F_1(x)=2(\arcsin\sqrt{x})^2+4\left(\frac{1-x}{x}\right)^{1/2}\arcsin\sqrt{x}-4\\ F_1(1)=3-\frac{1}{4}\pi^2\end{gathered} \tag{A5}$$

Using

$$\frac{d}{dx}[xF_2(x)]=x\frac{dF_1}{dx}$$

and integrating, we obtain

$$\begin{gathered}F_2(x)=3-\frac{(\arcsin\sqrt{x})^2}{x}-2\left(\frac{1-x}{x}\right)^{1/2}\arcsin\sqrt{x}\\ F_2(1)=3-\frac{1}{4}\pi^2\end{gathered} \tag{A6}$$

Finally,

$$\frac{x\,dF_3}{dx}=F_1(x)$$

so that

$$F_3(x) = 8 - 4(\arcsin\sqrt{x})^2 - 8\left(\frac{1-x}{x}\right)^{1/2} \arcsin\sqrt{x} + 4I_1(\arcsin\sqrt{x})$$

where

$$I_1(y) = \int_0^y z^2 \cot(z)\, dz \tag{A7}$$

One can either use the integral as the definition of $I_1(y)$ or rewrite it in terms of the logarithmic integral functions $li_2(y)$ and $li_3(y)$,

$$I_1(y) = -2i\frac{y^3}{3} + y^2(i\pi + \ln 2) + iyli_2(e^{-2ix}) - li_3(e^{-2ix}) \tag{A8}$$

where

$$li_n(x) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

We notice that

$$S_{1,1}(T=0) = \frac{3\pi^2}{4} - 7 \tag{A9}$$

Next consider the sums

$$A = \sum_{k=1}^{\infty} \frac{R_{1k}}{k+1} = -2\sum_{k=1}^{\infty} \frac{4^k B(1+k, 1+k)}{k(k+1)} \tag{A10}$$

Using $B(1+k, 1+k) = \int_0^1 dt\, t^k(1-t)^k$, we obtain

$$A = 2\int_0^1 dt\, [\ln(1-x) - 1 - x^{-1}\ln(1-x)] \tag{A11}$$

where $x = 4t(1-t)$. After two subsequent changes of variables, $t = \sin^2\Theta$, and then $y = \cos 2\Theta$, we obtain

$$A = -2 - 4\int_0^1 dy \ln yy^2(1-y^2)^{-1} \tag{A12}$$

Expanding the denominator, integrating term by term, and summing, we obtain

$$A = -2 - 4\left[1 - \frac{3}{4}\zeta(2)\right] = \frac{\pi^2}{2} - 6 \tag{A13}$$

Next we need the functions

$$H_0(x) = \sum_{k=1}^{\infty} B(k,k)\, x^k, \qquad H_1(x) = \sum_{k=1}^{\infty} B(k,k)\, \frac{x^k}{k^2}$$
$$H_2(x) = \sum_{k=1}^{\infty} B(k,k)\, \frac{x^k}{k+1}, \qquad H_3(x) = \sum_{k=1}^{\infty} B(k,k)\, \frac{x^k}{k^2} \tag{A14}$$
$$H_4(x) = \sum_{k=1}^{\infty} B(k,k)\, \frac{x^k}{k^3}$$

Using the integral representation for $B(k,k)$, we have that

$$H_1(x) = -2 \int_0^{1/2} dy\, (y-y^2)^{-1} \ln[1 - x(y-y^2)] \tag{A15}$$

Changing variables via

$$y - y^2 = \frac{z^2}{4}$$

we obtain

$$H_1(x) = -4 \int_0^1 dz\, z^{-1}(1-z^2)^{-1/2} \ln\left(1 - x\frac{z^2}{4}\right)$$
$$= \left[\pi - 2\arccos\left(\frac{1}{2}\sqrt{x}\right)\right]^2 \tag{A16}$$
$$H_1(4) = \pi^2$$

Using

$$x\frac{dH_1(x)}{dx} = H(x)$$

we obtain

$$H_0(x) = \left(\frac{x}{4-x}\right)^{1/2} \left[2\pi - 4\arccos\left(\frac{1}{2}\sqrt{x}\right)\right] \tag{A17}$$

From

$$\frac{d}{dx}[xH_2(x)] = H_0(x)$$

we obtain, after integrating,

$$
\begin{aligned}
H_2(x) = {} & 2 + \frac{2\pi^2}{x} - 2\pi(4-x)^{1/2}\, x^{-1/2} + 4(4-x)^{1/2}\, x^{-1/2} \arccos\left(\frac{1}{2}\sqrt{x}\right) \\
& + \frac{16}{x} \arctan[(4-x)^{1/2}\, x^{-1/2}] \arccos\left(\frac{1}{2}\sqrt{x}\right) \\
& - \left(\frac{8}{x}\right) \arctan^2[(4-x)^{1/2}\, x^{-1/2}] \\
& - \left(\frac{8\pi}{x}\right) \arctan[(4-x)^{1/2}\, x^{-1/2}]
\end{aligned}
\tag{A18}
$$

$$
H_2(4) = \frac{\pi^2}{2} + 2
$$

For $H_3(x)$ we have

$$
x \frac{dH_3}{dx} = H_1
$$

so

$$
H_3(x) = \int_0^x \frac{dy}{y} \left[\pi - 2 \arccos\left(\frac{1}{2}\sqrt{y}\right)\right]^2 \tag{A19}
$$

This cannot be evaluated in terms of special functions and must be evaluated numerically. Similarly,

$$
H_4(x) = \int_0^x \frac{dy}{y} H_3(y) \tag{A20}
$$

## APPENDIX B. COMPARISON WITH OTHER EXPANSIONS

It is instructive to compare the calculation with three terms in the $\delta$ expansion to the calculation using many terms in the weak and strong coupling expansions. The simplest place to make the comparison is for the equal-time correlation functions because we can obtain these expansions directly from the integral (1.12),

$$
\langle x^2 \rangle_{\text{eq}} = \frac{\int_{-\infty}^{\infty} dx\, x^2 \tilde{P}(x)}{\int_{-\infty}^{\infty} dx\, \tilde{P}(x)}, \qquad \tilde{P}(x) = \exp\left[\frac{-mx^2}{2} + \frac{gx^{2(1+\delta)}}{2+2\delta}\right] \tag{B1}
$$

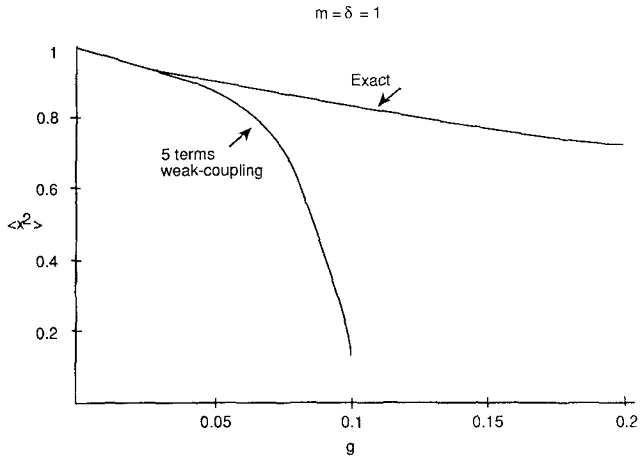


Fig. 10. Comparison of the first five terms in the weak coupling expansion (B.3) to the exact evaluation of $\langle x^2 \rangle$ in (3.5), for $m = \delta = 1$.

The weak coupling expansion for this is obtained by using the Gaussian part as the measure and expanding in powers of $g$ about that. That is, we write

$$\tilde{P}(x) = e^{-mx^2/2} \sum_{n=0}^{\infty} \left[ \frac{-g}{2+2\delta} \right]^n \frac{x^{2n(1+\delta)}}{n!} \tag{B2}$$

Performing the integrals, we obtain

$$\langle x^2 \rangle = \frac{2}{m} \frac{\sum_0^\infty [-g/(2+2\delta)]^n \, \Gamma(n+\delta n+3/2)(2/m)^{n+\delta n}/n!}{\sum_0^\infty [-g/(2+2\delta)]^n \, \Gamma(n+\delta n+1/2)(2/m)^{n+\delta n}/n!} \tag{B3}$$

The weak coupling expansion is then obtained by reexpanding this expression as a power series in $g$ starting with $g^0$. The weak coupling expansion is of course an asymptotic series.

The result of keeping five terms in the weak coupling expansion is compared in Fig. 10 with the exact numerical evaluation of (3.5). We notice that the weak coupling expansion breaks down at very small values of $g$. Taking a Padé approximant does not improve this result, as the denominators have poles at small values of $g$ preventing an effective analytic continuation to large $g$.

For the integral (3.5) one can also perform a strong coupling expansion by treating the Gaussian part as a perturbation. That is, we write

$$\tilde{P}(x) = \exp\left[ \frac{-gx^{2+2\delta}}{2+2\delta} \right] \sum_{n=0}^{\infty} \frac{(-mx^2/2)^n}{n!} \tag{B4}$$

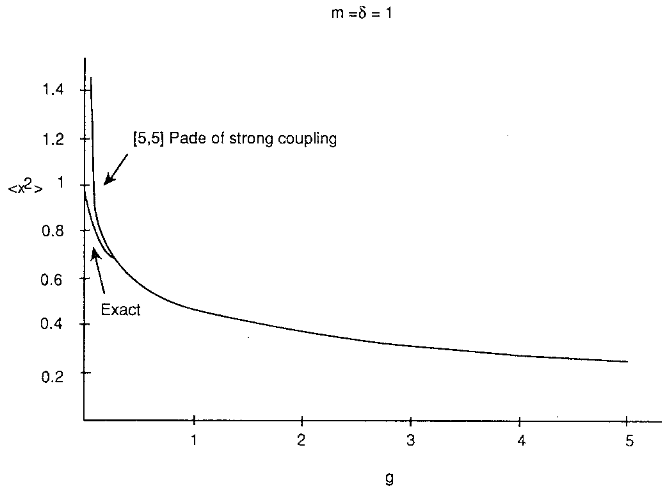


Fig. 11. The [5, 5] Padé approximant of the strong coupling expansion (B.5) compared to the exact evaluation of $\langle x^2 \rangle$ in (3.5), for $m = \delta = 1$.

Performing the integrals, we now obtain

$$\langle x^2 \rangle = \left(\frac{2+2\delta}{g}\right)^{1/(1+\delta)} \times \frac{\sum_{n=0}^{\infty}(-m^2/2)^n\,[(2+2\delta)/g]^{n/(1+\delta)}\,\Gamma((2n+3)/(2+2\delta))/n!}{\sum_{n=0}^{\infty}(-m^2/2)^n\,[(2+2\delta)/g]^{n/(1+\delta)}\,\Gamma((2n+1)/(2+2\delta))/n!} \tag{B5}$$

Reexpanding this in terms of powers of $(1/g)^{1/(1+\delta)}$, one obtains the strong coupling expansion. This expansion is *convergent.* Padé approximants to the strong coupling expansion are a very accurate representation of the answer except for quite small $g$, as seen in Fig. 11. Unlike the weak coupling expansion and the $\delta$ expansion, the strong coupling expansion exists only for zero-dimensional field theory (ordinary integrals as opposed to functional integrals). For dimensions greater than zero the strong coupling expansion must be regulated, for example, by introducing a lattice, because it is a singular perturbation theory in derivative terms. Thus, in higher dimensions the strong coupling expansion introduces a new parameter (the cutoff or lattice spacing) into the theory, which is present even after mass and coupling constant renormalizations.[9] In particular, a strong coupling expansion for the correlation functions (1.7) does not exist in the continuum. The lattice-regulated strong coupling expansion for the path integral (1.7) is discussed in ref. 6. On the other hand, the $\delta$ expansion, after being analytically continued using Padé approximants, works uniformly in both weak and strong coupling regimes and does not require the introduction of a lattice regulator in higher dimensions.

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